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Weights on the Real Line That Admit Good Relative Polynomial Approximation, with Applications

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Let $W(x)=\exp(-Q(x))$ be a weight on the real line, with Q satisfying conditions typically imposed by Freud. For large enough n, let q_n denote the positive root of the equation $q_nQ'(q_n)=n$. For a large class of weights of this type, we construct polynomials $P_n(x)$ of degree at most n, such that for n large enough, $P_n(x)\sim W(x)$, $|x|\leqslant Cq_n$, where $C>0$ is independent of n. We apply these to prove L_p Markov–Bernstein inequalities ($0<p\leqslant\infty$) that are new for $0<p<1$, except in special cases. Further applications include lower bounds for Christoffel functions that are new for weights such as $\exp(-|x|^\alpha(\log|x|)^\beta(\log\log|x|)^\gamma\ldots)$, $|x|$ large enough, if $1<\alpha<2$. © 1987 Academic Press, Inc.

1. INTRODUCTION

Let $\alpha>1$ and $W_\alpha(x)=\exp(-|x|^\alpha)$, $x\in\mathbb{R}$. In [6], we constructed polynomials $P_n(x)$, of degree at most n, $n=1, 2, 3,\ldots$, such that

$$P_n(x)\sim W_\alpha(x),\qquad |x|\leqslant Cn^{1/\alpha},$$

and

$$|P_n'(x)|\leqslant C_1n^{1-1/\alpha}W_\alpha(x),\qquad |x|\leqslant Cn^{1/\alpha},$$

where C and C_1 are independent of n and x. As usual, $\sim$ denotes that the ratio of the two functions is bounded above and below by positive con-

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stants independent of n and x. These polynomials were used to estimate Christoffel functions and to prove L_p Markov–Bernstein inequalities for $0<p\leqslant\infty$.

It is the purpose of this paper to extend the results of [6] to more general weights. To this end, we need two definitions:

DEFINITION 1.1. Let $W(x)=\exp(-Q(x))$, where Q is even and continuous in $\mathbb{R}$, and assume there exist A, $B>0$, and $0<\theta<1$ such that Q'' is continuous in $[A,\infty)$ and

$$Q'(u)>0, \qquad u\in[A,\infty), \tag{1.1}$$

$$-\theta\leqslant uQ''(u)/Q'(u)\leqslant B, \qquad u\in[A,\infty). \tag{1.2}$$

Then we shall call W a Freud weight. Associated with a Freud weight are the quantities q_n, defined to be the positive root of the equation

$$q_nQ'(q_n)=n, \tag{1.3}$$

for n large enough.

The above definition of a Freud weight is essentially the same as that in [9] and the existence and uniqueness of q_n follows from Lemma 7 in [9].

DEFINITION 1.2. Let W be a Freud weight. Assume there exist positive constants C_1 and C_2 independent of n and x, and even polynomials $P_n(x)$ of degree at most n, such that for n large enough

$$P_n(x)\sim W(x), \qquad |x|\leqslant C_1q_n, \tag{1.4}$$

$$|P_n'(x)|\leqslant C_2(n/q_n)\,W(x), \qquad |x|\leqslant C_1q_n. \tag{1.5}$$

Then we shall call W a regular weight.

We shall investigate conditions under which Freud weights are regular weights. Once the regularity of a weight is established, one can easily obtain lower bounds for the associated Christoffel functions (compare Freud [4], Nevai [10], and L_p Markov–Bernstein inequalities for $0<p\leqslant\infty$. For weights $W(x)=\exp(-Q(x))$, where $Q(x)$ grows at least as fast as x^2 as $|x|\to\infty$, the new elements of our work are the L_p Markov–Bernstein inequalities for $0<p<1$. The approach adopted by Freud [5] cannot deal with the case $0<p<1$, so we use an approach similar to that of Bonan and Nevai [3]. In the case where $Q(x)$ grows slower than x^2 as $|x|\to\infty$, the lower bounds for the Christoffel functions are new as well.

As examples of weights that we can prove to be regular, we mention

$$W(x)=(1+x^2)^{\Delta}\exp(-|x|^{\alpha}(\log|x|)^{\beta_1}(\log_2|x|)^{\beta_2}\cdots(\log_k|x|)^{\beta_k}), \tag{1.6}$$

where Δ is real, $\alpha > 1$, and $\beta_j \geqslant 0$, $j = 1, 2, \ldots, k$, while $\log_k$ denotes the kth iterated logarithm. Of course, the weight is defined by (1.6) only for $|x|$ large enough and must be suitably modified for small $|x|$. It is also possible to handle weights such as

$$W(x) = \exp(-|x|^{\alpha}(\log|x|)^{\beta}), \qquad \beta < 0. \tag{1.7}$$

provided $\alpha > 1$ but $\alpha \neq 2$. Unfortunately, for technical reasons relating to canonical products of integral order, we cannot deal with the weight (1.7) if $\alpha = 2$.

We now mention an important special case of our results that can deal with the weights in (1.6) and (1.7), except when $\alpha = 2$ or $\alpha = 4$.

THEOREM 1.3. *Let $W = \exp(-Q)$ be a Freud weight. Assume that*

$$\lim_{x \to \infty} xQ'(x)/Q(x) = \alpha. \tag{1.8}$$

where $\alpha > 1$, but $\alpha \neq 2$ and $\alpha \neq 4$. Then W is a regular weight.

As in [6], our basic idea is to use canonical products $G_{\phi}(z)$ of Weierstrass primary factors, with only negative real zeros. Much of the groundwork was laid in [6], and the main task of this paper is to determine when we can find ϕ such that, for example, $W = \exp(-Q) \sim |\exp((-1)^l H)|$ in the notation of [6, Lemma 3.3]. This is completed in Theorems 2.1 and 2.6. Using the entire functions constructed by one of us [9], we show in Theorems 2.4 and 2.5 that a very large class of Freud weights are regular.

The paper is organized as follows: In Section 2, we define our notation and state our main results. In Section 3, we state Markov–Bernstein inequalities, estimates of Christoffel functions, and so on. In Section 4, we prove Theorems 2.1 and 2.6 and in Section 5 we prove Theorem 2.4. Finally, in Section 6 we prove Theorem 2.5, and in Section 7 we prove Theorem 1.3.

2. Notation and Main Results

Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n and x. Different occurrences of the same symbol do not necessarily denote the same constant. When stating inequalities for polynomials P of degree at most n, the constants will be independent of P, n, and x. To denote dependence of constants C on parameters $\alpha, p, \ldots$, we write $C = C(\alpha, p)$, and so on. The usual symbols $\sim$, o, and O will be used to compare functions and sequences. Thus, $f(x) \sim g(x)$ if for some C_1 and C_2, $C_1 \leqslant f(x)/g(x) \leqslant C_2$ for all x considered.

Associated with a Freud weight $W=\exp(-Q)$, there is the quantity

$$\alpha=\alpha(W)=\limsup_{x\to\infty} \log Q(x)/\log(x). \tag{2.1}$$

It is shown in [9, Lemma 7(v)] that $\alpha \leqslant B+1$, where B is as in Definition 1.1. We note that the case $\alpha<1$ leads to an indeterminate moment problem (see Akhiezer [1, p. 87, Problem 14]). Further it was shown in [6] that the weights $\exp(-|x|^{\alpha})$, $0<\alpha\leqslant 1$, cannot be regular. Thus we restrict ourselves to the case $\alpha>1$. Even the cases $\alpha=2$ and $\alpha=4$ pose certain difficulties because of the delicate behavior of canonical products of integral order.

The case where $\alpha>2$ or $\alpha=2$ and $Q(t)\neq o(t^2)$, $|t|\to\infty$ is dealt with in Theorem 2.1, while Theorem 2.5 deals with the case $\alpha>2$, $\alpha\neq 4$. Theorem 2.6 may be used for the case $1<\alpha<2$, while Theorem 2.4 is a comparison theorem, which may be used for any $\alpha>1$.

THEOREM 2.1. *Let $W=\exp(-Q)$ be a Freud weight. Suppose that for some positive even integer k, Q admits the following representation*:

$$Q(r^{1/k})=\int_1^{\infty}\frac{\phi(t)}{t+r}\left(\frac{r}{t}\right)^2 dt+cr+g(r), \qquad r\in[0,\infty), \tag{2.2}$$

where ϕ is a function, positive, continuous, and increasing on $[1,\infty)$ such that

$$\int_1^{\infty}\phi(t)\,t^{-3}dt<\infty \qquad but \qquad \int_1^{\infty}\phi(t)\,t^{-2}dt=\infty, \tag{2.3}$$

while $c\in(-\infty,\infty)$ and $g(r)$ is a bounded real function. Then W is a regular weight.

Note that if we add a bounded function to Q, then the new weight $\sim$ the old one. Thus we may assume that (2.2) holds only for r large enough. Further, since

$$\frac{1}{t+r}\left(\frac{r}{t}\right)^2=\frac{r}{t^2}-\frac{1}{t}+\frac{1}{t+r}, \tag{2.4}$$

it follows from (2.2) that in changing ϕ on a finite interval we change only the remainder terms in (2.2). Thus we need only assume that $\phi(t)$ is positive and increasing for large enough t. For the same reason, we may replace the lower limit of the integral in (2.2) by any positive number. At this stage, it is pertinent to discuss two examples.

EXAMPLE 2.2. Let

$$Q(x)=|x|^{\alpha}(\log|x|)^{\beta}, \qquad |x|\geqslant 2, \tag{2.5}$$

where $\alpha\geqslant 2$ and $\beta\geqslant 0$, and if $\alpha=2$ or $\alpha=4$, then $\beta>0$. Further define a positive even integer k by

$$k=\begin{cases} 2^m & \text{if } 2^m<\alpha<2^{m+1}, \text{ some } m\geqslant 1,\\ 2^m-2 & \text{if } \alpha=2^m, \text{ some } m\geqslant 3,\\ \alpha & \text{if } \alpha=2 \text{ or } 4. \end{cases} \tag{2.6}$$

Then $\alpha'=\alpha/k$ satisfies

$$1<\alpha'<2 \qquad \text{if } \alpha>2, \alpha\neq 4,$$
$$\alpha'=1 \qquad \text{if } \alpha=2, \alpha=4.$$

Consider the function

$$f(z)=(-z)^{\alpha'}(\log(-z))^{\beta}, \qquad z\in\mathbb{C}\backslash[-1,\infty), \tag{2.7}$$

where the powers and logs have their principal values. Let $f(t+i0)=\lim_{\varepsilon\to 0+}f(t+i\varepsilon)$, $t\in\mathbb{R}$. A straightforward, but tedious calculation shows that

$$\operatorname{Im} f(t+i0)$$
$$=\begin{cases} 0, & t<-1,\\ |t|^{\alpha'}|\log|t||^{\beta}\sin(-\beta\pi), & -1\leqslant t\leqslant 0,\\ |t|^{\alpha'}\{(\log t)^2+\pi^2\}^{\beta/2}\sin(-\pi\alpha'-\beta\arctan(\pi/\log t)), & t>0. \end{cases} \tag{2.8}$$

A similar representation holds for $\operatorname{Re} f(t+i0)$. We observe that $\operatorname{Re} f(t+i0)$ and $\operatorname{Im} f(t+i0)$ are differentiable in $\mathbb{R}$, except at $-1, 0$, and 1. Defining $f(t-i0)$ in a similar manner, we see that

$$f(t-i0)=\overline{f(t+i0)}, \qquad t\in\mathbb{R}. \tag{2.9}$$

We now use contour integral techniques. Let $s>2$. Let Γ_2 and Γ_s denote circles centred on 0, of radius 2 and s, respectively, oriented as in Fig. 1. Let $L_{\pm}=\{t\pm i0\colon t\in[2,s]\}$, oriented as in Fig. 1, so that $\Gamma=\Gamma_2\cup L_+\cup\Gamma_s\cup L_-$ is a closed curve. Since $f(t+i0)$ and $f(t-i0)$ are continuous in $[2,s]$, Cauchy's integral formula shows that for $-z$ inside Γ,

$$f(-z)/z^2=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{t+z}\frac{dt}{t^2}.$$

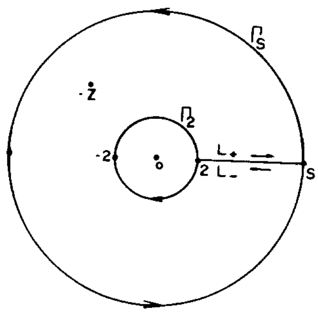

FIGURE 1

Since $f(t)/t^2$ vanishes at ∞ (uniformly in arg t), we see that

$$\int_{\Gamma_s} \frac{f(t)}{t+z}\frac{dt}{t^2} \to 0, \qquad \text{as } s \to \infty,\ z \text{ fixed.}$$

Hence, for $-z$ outside $\Gamma_2 \cup [2, \infty)$,

$$\frac{f(-z)}{z^2} = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(t)}{t+z}\frac{dt}{t^2} + \frac{1}{2\pi i}\int_2^\infty \frac{f(t+i0)-f(t-i0)}{t+z}\frac{dt}{t^2},$$

and by (2.4) and (2.9),

$$\begin{aligned}\operatorname{Ref}(-z) = \operatorname{Re}\left\{\frac{1}{2\pi i}\int_{\Gamma_2} f(t)\left\{\frac{z}{t^2} - \frac{1}{t} + \frac{1}{t+z}\right\} dt\right\} \\ + \frac{1}{\pi}\int_2^\infty \frac{\operatorname{Imf}(t+i0)}{t^2} \operatorname{Re}\left\{\frac{z^2}{t+z}\right\} dt. \qquad (2.10)\end{aligned}$$

In particular, as $f(-r)$ is real for $r \in (3, \infty)$,

$$f(-r) = c_1 r + g_1(r) + \frac{1}{\pi}\int_2^\infty \frac{\operatorname{Imf}(t+i0)}{t+r}\left(\frac{r}{t}\right)^2 dt, \qquad (2.11)$$

where c_1 is real, $g_1(r)$ is bounded, and both are defined in an obvious way from (2.10). Next, let

$$\phi(t) = \{\pi k^\beta\}^{-1} \operatorname{Imf}(t+i0), \qquad t \in (2, \infty).$$

We see from (2.8), by differentiation, that $\phi(t)$ is positive and increasing for large t—note that $\beta > 0$ if $\alpha' = 1$. Further, as $1 \leqslant \alpha' < 2$, $\phi(t)$ obviously satifies (2.3). Finally, by (2.5) and (2.7),

$$Q(r^{1/k}) = k^{-\beta} f(-r), \qquad r \in (3, \infty),$$

and together with (2.11) this essentially yields (2.2). The remarks after Theorem 2.1 show that the different range of integration and range of r are not important. Hence $W = \exp(-Q)$ is a regular weight.

EXAMPLE 2.3. Let $Q(x)$ be given by (2.5), where $\alpha > 2$ and $\beta < 0$. Define a positive even integer k by

$$k = \begin{cases} 2^m & \text{if } 2^m < \alpha < 2^{m+1}, \quad \text{some } m \geq 1 \\ 2^m - 2 & \text{if } \alpha = 2^m, \quad \text{some } m \geq 2. \end{cases}$$

Then $\alpha' = \alpha/k$ satisfies

$$1 < \alpha' < 2 \qquad \text{if } \alpha > 2, \alpha \neq 4,$$

$$\alpha' = 2 \qquad \text{if } \alpha = 4.$$

Let $f(z)$ be as in (2.7). We see that (2.8)–(2.11) hold as before, even if $\alpha' = 2$, as $f(-t)/t^2$ still vanishes at ∞. We can define $\phi(t)$ exactly as before. If $1 < \alpha' < 2$, (2.3) follows, while if $\alpha' = 2$, (2.3) still follows, since from (2.8),

$$\phi(t) = O(t^2(\log t)^\beta(\log t)^{-1}), \qquad t \to \infty.$$

Finally, (2.2) follows as before, and so $W = \exp(-Q)$ is a regular weight.

In much the same way, we can deal with all the weights in (1.6) if $\alpha \geq 2$, though the more complicated the weight is, the more difficult it is to check the representation. The case $1 < \alpha < 2$ will be discussed after Theorem 2.6. Note that the integral in (2.2) may be analytically continued to $\mathbb{C} \backslash (-\infty, -1)$, and hence the functions Q in Theorem 2.1 are analytic, apart from an essential bounded function. The following "comparison theorem" enables us to consider non-analytic weights.

THEOREM 2.4. *Let $W = \exp(-Q)$ be a regular weight and assume $Q''(x)$ is positive for large enough x, while $Q'(\infty) = \infty$. Let $W_1 = \exp(-Q_1)$ be a Freud weight such that*

$$Q_1(x) \sim Q(x), \qquad x \text{ large enough}, \tag{2.12}$$

and

$$|Q_1''(x)| \leq CQ''(x), \qquad x \text{ large enough}. \tag{2.13}$$

Then W_1 is also a regular weight.

It is noteworthy that even if Q_1 does not satisfy the conditions (1.1) and (1.2) in Definition 2, the proof of Theorem 2 nevertheless shows that we can find polynomials satisfying (1.4) and (1.5) for W_1. Using Theorems 1 and 2, we shall prove

THEOREM 2.5. *Let $W=\exp(-Q)$ be a Freud weight, and let $\alpha(W)>2$, but $\alpha(W)\neq 4$. Let*

$$k=\begin{cases}2^m & \text{if } 2^m<\alpha<2^{m+1}, \quad \text{some } m\geqslant 1,\\ 2^m-2 & \text{if } \alpha=2^m, \quad \text{some } m\geqslant 3.\end{cases} \tag{2.14}$$

Suppose there exist constants a, b such that

$$2^k<a\leqslant Q(2x)/Q(x)\leqslant b<2^{2k}, \qquad x \text{ large enough.} \tag{2.15}$$

Then W is a regular weight.

There are a number of simple conditions that imply the rather cumbersome (2.15). For example, if $Q(x)=|x|^\alpha T(|x|)$, where $T(2x)/T(x)\to 1$, $x\to\infty$, then (2.15) holds. Similarly, if for some $1<a'<b'<2$, we have

$$a'\leqslant uQ'(u)/(kQ(u))\leqslant b', \qquad u \text{ large enough,} \tag{2.16}$$

then one check that (2.15) is valid by multiplying both sides of (2.16) by $1/u$, and integrating from x to $2x$. In particular, (2.16) is valid if

$$\lim_{u\to\infty} uQ'(u)/Q(u)=\alpha(W), \tag{2.17}$$

and so Theorem 2.5 implies Theorem 1.3 in the case $\alpha(W)>2$, $\alpha(W)\neq 4$.

THEOREM 2.6. *Let $W=\exp(-Q)$ be a Freud weight. Suppose that Q admits the following representation:*

$$Q(r^{1/2})=-\mathrm{Re}\left\{\int_1^\infty \frac{\phi(t)}{t+z}\left(\frac{z}{t}\right)dt\right\}+g(r), \qquad r\in[0,\infty), \tag{2.18}$$

where $g(r)$ is a bounded real function, $z=r\exp(i\theta_0)$, $\theta_0\in(-\pi,\pi)$, and ϕ is a function, positive and differentiable for large enough t, with

$$\phi'(t)>0, \qquad t \text{ large enough} \tag{2.19}$$

and for some $1<\alpha<2$,

$$\lim_{t\to\infty} t\phi'(t)/\phi(t)=\alpha/2. \tag{2.20}$$

Assume also

$$\cos(\alpha\theta_0/2)<0. \tag{2.21}$$

Then W is a regular weight.

The remarks made after Theorem 2.1 about changing Q or ϕ in a bounded interval apply to Theorem 2.6 as well. It is noteworthy that with much extra effort, the restriction (2.20) can be somewhat weakened. In any event, Theorem 2.6 will be sufficiently powerful to prove Theorem 1.3 in the case $1 < \alpha(W) < 2$, and in turn, Theorem 1.3 applies to all the weights (1.6) and (1.7) if $1 < \alpha < 2$.

Note, finally, that if W and W_1 are regular weights, so is WW_1. Further, one can use Theorem 2.4 to show that for any $r > 0$, W^r is regular.

3. Inequalities for Regular Weights

We first need some properties of Freud weights:

LEMMA 3.1. *Let W be a Freud weight, and θ and B be as in Definition* 1.1.

(i) $xQ'(x)$ *is increasing for large* x.

(ii) $2^{1/(1+B)} \leqslant q_{2n}/q_n \leqslant 2^{1/(1-\theta)}$ *for n large enough.*

(iii) $C_1 x^{-\theta} \leqslant Q'(x) \leqslant C_2 x^B$ *for x large enough.*

(iv) $C_3 x^{1-\theta} \leqslant Q(x) \leqslant C_4 x^{1+B}$ *for x large enough.*

(v) $Q(x) \sim xQ'(x)$ *for x large enough.*

(vi) $\limsup_{n \to \infty} Q(q_n)/n < \infty$.

(vii) $C_5 n^{1/(1+B)} \leqslant q_n \leqslant C_6 n^{1/(1-\theta)}$ *for n large enough.*

Proof. These are all proved in Lemma 7 in [9] except for (ii), which may easily be deduced from (i) and (iii) of Lemma 7 in [9]. ∎

Next, we need an infinite–finite range inequality:

LEMMA 3.2. *Let W be a Freud weight and let $0 < p \leqslant \infty$. There exist n_0, C_1, and C_2 depending on W and p only, such that for every polynomial P of degree at most n, $n \geqslant n_0$,*

$$\|PW\|_{L_p(\mathbb{R})} \leqslant C_1 \|PW\|_{L_p(-C_2 q_n, C_2 q_n)}. \tag{3.1}$$

Proof. As W is even and decreasing in (A, ∞), we may apply Theorem A in Lubinsky [7] with $g = 1$ to deduce (3.1), but with $(-C_2 q_n, C_2 q_n)$ replaced by $(-11 q_{2n}, 11 q_{2n})$. In view of Lemma 3.1(ii), (3.1) follows as stated. ∎

We can now prove Markov–Bernstein inequalities. Note that n is restricted below, only in order that q_n be defined.

THEOREM 3.3 (Local Markov–Bernstein inequality). *Let W be a regular weight. Let $0<p\leqslant\infty$. Let $0<\eta<\xi<\infty$. There exist n_0 and C depending on W, η, ξ, and p only, such that for all polynomials P of degree at most n, $n\geqslant n_0$,*

$$\|P'W\|_{L_p(-\eta q_n,\,\eta q_n)}\leqslant C(n/q_n)\|PW\|_{L_p(-\xi q_n,\,\xi q_n)}.$$

Proof. This is exactly the same as that of Theorem 7.3 in [6]. ∎

COROLLARY 3.4 (Global Markov–Bernstein inequality). *Let W be a regular weight. Let $0<p\leqslant\infty$. There exist n_0 and C depending on W and p only, such that for all polynomials P of degree at most n, $n\geqslant n_0$,*

$$\|P'W\|_{L_p(\mathbb{R})}\leqslant C(n/q_n)\|PW\|_{L_p(\mathbb{R})}.$$

Proof. This follows from Lemma 3.2 and Theorem 3.3. ∎

Except for the weights $\exp(-|x|^\alpha)$, $\alpha>1$, Corollary 3.4 is new when $0<p<1$. See [6] for further discussion and references. Next, we estimate Christoffel functions:

THEOREM 3.5. *Let W be a regular weight, and assume further that*

$$Q''(x)\geqslant 0,\qquad x \text{ large enough}, \tag{3.2}$$

and for some C

$$Q'(2x)/Q'(x)>C>1,\qquad x \text{ large enough}. \tag{3.3}$$

Let $0<p\leqslant\infty$ and j be a nonnegative integer. For $n=j+1, j+2,\ldots$, define

$$\lambda_{n,p}(W,j,x)=\inf\|PW\|_{L_p(\mathbb{R})}/|P^{(j)}(x)|,\qquad x\in\mathbb{R},$$

where the infimum is over all polynomials P of degree at most $n-1$. Then there exist C_1 and C_2 depending only on j, p, and W such that

(i) $\lambda_{n,p}(W,j,x)\sim(q_n/n)^{j+1/p}W(x)$, $|x|\leqslant C_1q_n$,
(ii) $\lambda_{n,p}(W,j,x)\geqslant C_2(q_n/n)^{j+1/p}W(x)$, $x\in\mathbb{R}$.

Proof. The lower bound in (ii) may be proved in exactly the same way as Theorem 7.4 in [6] and is valid even when we do not assume (3.2) and (3.3). To obtain the matching upper bounds needed for (i), we apply Theorem 3.4 in [8]. To this end, we must first verify (3.10), (3.11), and (3.12) in [8]. First, from (3.2) above it follows that the left member of (3.10) in [8] is identically zero, and hence (3.10) in [8] holds trivially. Next, as $Q'(x)$ is non-decreasing for large x (by (3.2)), $M_1(x)=Q'(x)$ for large x, and so (3.3) above implies (3.11) in [8]. Finally, to verify (3.12) in

[8], we note that if $x \geqslant \xi \geqslant C_1$, Lemma 3.1(v) and monotonicity of Q' show

$$Q(x) \geqslant CxQ'(x) \geqslant CxQ'(\xi).$$

Hence,

$$\begin{aligned} &3\xi Q'(\xi)\{\log(|x|/\xi)\}/Q(x) \\ &\quad \leqslant (3/C)(\xi/x)\log(x/\xi) < 1, \qquad \text{if } x/\xi \geqslant C_2,\ C_2 \text{ large enough.} \end{aligned}$$ ∎

COROLLARY 3.6. *Let W be a regular weight and satisfy* (3.2) *and* (3.3). *Let $p_k(W^2; x)$, $k = 0, 1, 2,\ldots$ be the orthogonal polynomials associated with the weights W^2, so that*

$$\int_{-\infty}^{\infty} p_k(W^2; x)\, p_j(W^2; x)\, W^2(x)\, dx = \begin{cases} 1, & k = j \\ 0, & k \neq j. \end{cases}$$

Let j be a nonnegative integer. Then for $n = j+1, j+2,\ldots,$

$$\sum_{k=0}^{n-1} (p_k^{(j)}(W^2; x))^2\, W^2(x) \begin{cases} \sim (n/q_n)^{2j+1}, & |x| \leqslant C_1 q_n, \\ \leqslant C_2 (n/q_n)^{2j+1}, & x \in \mathbb{R}. \end{cases}$$

One can also prove analogs of Theorem 7.6 and Theorem 7.7 in [6], and one can estimate the coefficients in the recurrence relation. Hence, a quantitative approximation theory may be developed for any regular weight.

4. PROOF OF THEOREMS 2.1 AND 2.6

The proofs of both Theorems 2.1 and 2.6 use results from [6], which we summarize in the following lemma. Recall the definition of the Weierstrass primary factor:

$$E(z, l) = \begin{cases} (1 - z), & l = 0, \\ (1 - z)\exp(z + z^2/2 + \cdots + z^l/l), & l \geqslant 1. \end{cases}$$

LEMMA 4.1. *Let $\phi^*(t)$ be a function, nonnegative, continuous, and nondecreasing in* $(0, \infty)$ *with* $\phi^*(1) = 1$. *Assume further that*

$$\lim_{t \to \infty} \phi^*(t) = \infty, \tag{4.1}$$

and assume there exists a nonnegative integer l such that

$$\int_1^{\infty} \phi^*(t)/t^{l+2} dt < \infty. \tag{4.2}$$

Let

$$H^*(z) = \int_1^{\infty} \frac{\phi^*(t)}{t + z} \left(\frac{z}{t}\right)^{l+1} dt, \qquad z \in \mathbb{C} \backslash (-\infty, 0), \tag{4.3}$$

and let c^* *be real, but* $c^*=0$ *if* $l=0$. *Further, let*

$$y(z)=\exp(2(-1)^l H^*(z)+2c^*z), \qquad z\in\mathbb{C}\backslash(-\infty,0). \tag{4.4}$$

Let ξ_n, $n=1,2,3,\ldots$ *be positive numbers such that for some* C_1 *and some* $\eta>1$,

$$1\leqslant\xi_n\leqslant C_1 n^{1/l}, \qquad n=1,2,3,\ldots \quad \text{if } l>0, \tag{4.5}$$

$$1\leqslant\xi_n\leqslant C_1 n^2/(\log n)^{2\eta}, \qquad n=1,2,\ldots \quad \text{if } l=0, \tag{4.6}$$

and

$$H^*(\xi_n)\leqslant C_1 n, \qquad n=1,2,\ldots. \tag{4.7}$$

Let $-\pi<\theta_0<\pi$, *and write* $z=r\exp(i\theta_0)$, $r\in[0,\infty)$. *There exist polynomials* $T_n(z)$ *of degree at most* $4n$, $n=1,2,3,\ldots$, *and* C_n *and* C_3 *such that*

$$T_n(r)\sim|y(z)|, \qquad r\in[0, C_2\xi_n], \tag{4.8}$$

and

$$|T_n'(r)/y(z)|\leqslant\begin{cases}C_3H^*(r)/r, & r\in[1,C_2\xi_n],\\ C_3(H^*(r)/r+1), & r\in(0,1].\end{cases} \tag{4.9}$$

Proof. Let r_n be the smallest root of the equation $\phi^*(r_n)=n$, $n=1,2,\ldots$ and let

$$G_{\phi^*}(z)=\prod_{n=1}^{\infty}E(-z/r_n,l), \qquad z\in\mathbb{C}.$$

It is shown in [6, Lemma 3.3] that G_{ϕ^*} is entire and

$$G_{\phi^*}(z)=\exp((-1)^l H^*(z)+U(z)+F(z)), \qquad z\in\mathbb{C}\backslash(-\infty,0),$$

where $H^*(z)$ is given by (4.3), $U(z)$ is a polynomial of degree at most l, and $F(z)$ is given by (3.13) in [6]. If $U^*(z)=U(z)-c^*z$, then U^* is still a polynomial of degree at most l, and

$$y(z)=\{G_{\phi^*}(z)\exp(-U^*(z))\exp(-F(z))\}^2, \qquad z\in\mathbb{C}\backslash(-\infty,0). \tag{4.10}$$

Lemma 5.2 in [6] shows that we can find polynomials $R_n(z)$ of degree at most n, $n=1,2,3,\ldots$, such that if $z=r\exp(i\theta_0)$,

$$|R_n(z)|\sim|G_{\phi^*}(z)\exp(-U^*(z))|, \qquad r\in(0,C_2\xi_n], \tag{4.11}$$

and

$$|R_n'(z)/\{G_{\phi^*}(z)\exp(-U^*(z))\}| \leqslant \begin{cases} C_3 H^*(r)/r, & r\in[1, C_2\xi_n] \\ C_3(H^*(r)/r+1), & r\in(0, 1]. \end{cases} \tag{4.12}$$

We note that although Lemma 5.2 in [6] is proved for $G_{\phi^*}\exp(-U)$, it holds with U replaced by U^* as the only property of U used in [6] was that U has degree at most l. Next, Proposition 4.1 in [6] shows that there exist polynomials $P_n(z)$ of degree at most n, $n=1, 2, 3, \ldots$, such that if $z=r\exp(i\theta_0)$,

$$|P_n(z)| \sim |\exp(-F(z))|, \tag{4.13}$$

and

$$|P_n'(z)| \leqslant C(1+|z|)^{-1}|\exp(-F(z))|, \tag{4.14}$$

for $r\in[0, \xi_n]$. Let

$$V_n(z)=P_n(z)\,R_n(z), \qquad n=1, 2, 3, \ldots$$

and

$$T_n(x)=|V_n(x\exp(i\theta_0))|^2, \qquad n=1, 2, 3, \ldots, x \text{ real},$$

so that V_n and T_n are polynomials of degree at most $2n$ and $4n$, respectively. Then (4.8) follows easily from (4.10), (4.11), and (4.13). Further, (4.9) follows from (4.10), (4.12), and (4.14)—compare the proof of Theorem 1 in [6, Sect. 6] and note that $H^*(r)$ is positive and nondecreasing [6, Lemma 5.1(iv)]. ∎

LEMMA 4.2. *Assume the hypothesis of Theorem* 1, *and let*

$$H(z)=\int_1^\infty \frac{\phi(t)}{t+z}\left(\frac{z}{t}\right)^2 dt, \qquad z\in\mathbb{C}\backslash(-\infty, 0). \tag{4.15}$$

Further, let ξ_n denote the positive root of the equation

$$H(\xi_n)=n, \qquad n=1, 2, 3, \ldots. \tag{4.16}$$

Then,

(i) $Q(r^{1/k})\sim H(r)$, *r large enough.*

(ii) $\xi_n\sim q_n^k$, *n large enough.*

Proof. We note that (4.15) coincides with the case $l=1$ of (3.12) in [6, Lemma 3.3]. Hence Lemma 5.1(iii), (iv), (v) in [6] are valid. In particular, $H(r)$ is strictly increasing, and so ξ_n is uniquely defined by (4.16).

(i) Note that

$$H(r)/r \geqslant \int_1^r \frac{\phi(t)\, r}{(t+r)\, t^2}\, dt \geqslant (1/2) \int_1^r \phi(t)/t^2\, dt$$

$$\to \infty, \qquad \text{as } r \to \infty, \tag{4.17}$$

by (2.3). Thus for large r, $H(r)$ is the dominant term in the right member of (2.2).

(ii) By (i) above and Lemma 3.1(v),

$$\xi_n^{1/k} Q'(\xi_n^{1/k}) \sim Q(\xi_n^{1/k}) \sim H(\xi_n) = n.$$

As $uQ'(u)$ is strictly increasing (Lemma 3.1(i)), it follows from (1.3) that there exist C_1 and C_2 such that

$$q_{[C_1 n]} \leqslant \xi_n^{1/k} \leqslant q_{[C_2 n]},$$

where $[x]$ denotes the integer part of x. The result now follows from Lemma 3.1(ii). ∎

Proof of Theorem 2.1. We shall apply Lemma 4.1 with $l=1$, $\theta_0=0$, and with a suitable choice of ϕ^* and c^*. Let

$$\phi^*(t) = \phi(t)/2, \qquad t \in [1, \infty).$$

The remarks after Theorem 2.1 show that we can alter ϕ in a finite interval so that $\phi^*(1)=1$. Then (2.3) immediately yields (4.2), while (2.3) and the monotonicity of ϕ yield (4.1). If H^* and H are defined by (4.3) and (4.15), respectively, we see that

$$H^*(z) = H(z)/2.$$

Let $c^* = -c/2$ and ξ_n be defined by (4.16). Now from (4.17), $n/\xi_n \to \infty$ as $n \to \infty$, and so (4.5) follows. Then Lemma 4.1 shows that there exist polynomials T_n of degree at most $4n$ for which (4.8) and (4.9) hold. Further, by (4.4) and as $\theta_0=0$,

$$|y(z)| = y(r) = \exp(-H(r) - cr)$$

$$= \exp(-Q(r^{1/k}) + g(r)) \sim \exp(-Q(r^{1/k})), \tag{4.18}$$

by (2.2). Let

$$P_n(x) = T_n(x^k), \qquad n = 1, 2, \ldots,$$

so that P_n has degree at most $4nk$. By (4.8) and (4.18),

$$P_n(x) \sim W(x)$$

if $x^k \leqslant C_2 \xi_n$, which is true if $|x| \leqslant Cq_n$, by Lemma 4.2(ii). Further, for $|x| \leqslant Cq_n$, (4.9) and the monotonicity of $H^*(r)/r$ (see [6, Lemma 5.1(iv)]) show that

$$\begin{aligned} |P_n'(x)| = k|x|^{k-1} |T_n'(x^k)| &\leqslant C_4 q_n^{k-1} (n/\xi_n) W(x) \\ &\leqslant C_5 n/q_n W(x). \end{aligned}$$

Finally, we can replace P_n by $P_{[n/(4k)]}$ to obtain polynomials of degree at most n, and can use

$$q_{[n/(4k)]} \sim q_n. \quad \blacksquare$$

We next proceed to the proof of Theorem 2.6.

LEMMA 4.3. *Assume that* Q, θ_0, α, *and* ϕ *satisfy* (2.18) *to* (2.21). *Let* $\alpha' = \alpha/2$. *Then*

(i) $\lim_{r \to \infty} \phi(rt)/\phi(r) = t^{\alpha'}, \; t \in (0, \infty).$ (4.19)

(ii) $\lim_{t \to \infty} \log \phi(t)/\log t = \alpha'.$ (4.20)

(iii) $\lim_{r \to \infty} Q(r)/\phi(r^2) = C > 0.$ (4.21)

(iv) *If* $g(r) \equiv 0$ *in* (2.18), *then*

$$\lim_{r \to \infty} r^j Q^{(j)}(r)/Q(r) = C_j > 0, \qquad j = 1, 2, \tag{4.22}$$

and consequently $Q'(\infty) = \infty$ *and* $Q''(x) \geqslant 0$ *for large enough* x.

Proof. (i) We prove more than (4.19) since we need more below. Let $\varepsilon > 0$. By (2.20), there exists $A = A(\varepsilon) > 0$ such that

$$(\alpha'/u)(1-\varepsilon) \leqslant \phi'(u)/\phi(u) \leqslant (\alpha'/u)(1+\varepsilon), \qquad u \geqslant A.$$

If $r \geqslant A$ and $t \geqslant 1$, we may integrate from r to rt:

$$\alpha'(1-\varepsilon) \log t \leqslant \log \phi(rt)/\phi(t) \leqslant \alpha'(1+\varepsilon) \log t.$$

If $t < 1$, but $rt \geqslant A$, we may integrate from rt to r:

$$\alpha'(1-\varepsilon) \log t \geqslant \log \phi(rt)/\phi(t) \geqslant \alpha'(1+\varepsilon) \log t.$$

Hence, if $r>0$ and $t\geqslant A/r$,

$$(\max\{t, 1/t\})^{-\alpha'\varepsilon}\leqslant\phi(rt)/(\phi(r)\,t^{\alpha'})\leqslant(\max\{t, 1/t\})^{\alpha'\varepsilon}. \tag{4.23}$$

It is now fixed, we can let $r\to\infty$ and use the fact that ε is arbitrary to deduce (4.19).

(ii) This follows easily by integrating (2.20).

(iii) Let

$$H(z)=\int_1^\infty \frac{\phi(t)}{z+t}\left(\frac{z}{t}\right)dt, \qquad z\in\mathbb{C}\backslash(-\infty, 0) \tag{4.24}$$

and

$$H_1(z)=\int_1^\infty \frac{t^{\alpha'}}{z+t}\left(\frac{z}{t}\right)dt, \qquad z\in\mathbb{C}\backslash(-\infty, 0). \tag{4.25}$$

By a well-known identity (Boas [2, p. 56, (4.1.6)]),

$$H_1(z)=(\pi/\sin\pi\alpha')\,z^{\alpha'}, \qquad z\in\mathbb{C}\backslash(-\infty, 0]. \tag{4.26}$$

Now, let $|\theta|<\pi$, $r\in(0,\infty)$, $z=re^{i\theta}$, and $w=e^{i\theta}$. Setting $t=ur$, we see

$$H(z)/\phi(r)=\int_{1/r}^\infty \frac{\phi(ru)}{\phi(r)}\frac{w}{u+w}\frac{du}{u}. \tag{4.27}$$

Unfortunately, because of a problem at $u=0$, we cannot apply Lebesgue's dominated theorem directly to this last integral. So, let $\varepsilon>0$ and $A=A(\varepsilon)$ be as in (4.23). Write

$$H(z)/\phi(r)=\left(\int_{1/r}^{A/r}+\int_{A/r}^\infty\right) \qquad (=I_1+I_2,\text{ say}).$$

We see that

$$|I_1|\leqslant\frac{\phi(A)}{\phi(r)}\int_{1/r}^{A/r}\frac{1}{|u+w|}\frac{du}{u}\leqslant(1-A/r)^{-1}\,\phi(A)\log A/\phi(r)\to 0 \text{ as } r\to\infty.$$

Further, for $u\in(A/r,\infty)$ (4.23) shows that the integrand in (4.27) is bounded by

$$u^{\alpha'}(\max\{u, 1/u\})^{\alpha'\varepsilon}|u+w|^{-1}\,u^{-1},$$

which is integrable over $(0,\infty)$ if ε is small enough, since $\alpha'\in(\frac{1}{2}, 1)$. Hence we can apply (4.19) and dominated convergence to I_2 to deduce that for $z = rw$, $w = e^{i\theta}$, and $|\theta| < \pi$,

$$\lim_{r \to \infty} H(z)/\phi(r) = \int_0^\infty \frac{u^{\alpha'}}{u+w} \left(\frac{w}{u}\right) du = H_1(w). \tag{4.28}$$

Now, let $\theta = \theta_0$. We see from (2.18), (4.26), and (4.28), that

$$\lim_{r \to \infty} Q(r^{1/2})/\phi(r) = -(\pi/\sin \pi\alpha') \cos \alpha'\theta_0. \tag{4.29}$$

Then (2.21) yields (4.21).

(iv) We first show that for $z = re^{i\theta}$, $r \in [0, \infty)$, $|\theta| < \pi$,

$$\lim_{r \to \infty} r^j H^{(j)}(z)/\phi(r) = H_1^{(j)}(w), \qquad j = 1, 2, \tag{4.30}$$

where $w = e^{i\theta}$. From (4.24),

$$rH'(z)/\phi(r) = (r/\phi(r)) \int_1^\infty \frac{\phi(t)}{(t+z)^2} dt$$

$$= \int_{1/r}^\infty \frac{\phi(ur)}{\phi(r)} \frac{du}{(u+w)^2}.$$

As $\phi(ur)/\phi(r) \leqslant 1$, $u \leqslant 1$, we can directly apply dominated convergence and (4.19) to deduce

$$\lim_{r \to \infty} rH'(z)/\phi(r) = \int_0^\infty u^{\alpha'}(u+w)^{-2} du = H_1'(w).$$

This proves (4.30) for $j = 1$. The case $j = 2$ is similar. Now let $\theta = \theta_0$ and $w = \exp(i\theta_0)$. As $g(r) \equiv 0$ in (2.18),

$$Q'(r) = -\text{Re}\{H'(r^2 w)\, 2rw\},$$

and so by (4.26) and (4.30)

$$\lim_{r \to \infty} rQ'(r)/\phi(r^2) = -\text{Re}\{H_1'(w)\, 2w\} = -2\alpha'(\pi/\sin \pi\alpha') \cos \alpha'\theta_0.$$

Similarly,

$$\lim_{r \to \infty} r^2 Q''(r)/\phi(r^2) = -\text{Re}\{H_1''(w)(2w)^2 + H_1'(w)(2w)\}$$

$$= -2\alpha'(2\alpha' - 1)(\pi/\sin \pi\alpha') \cos \alpha'\theta_0 > 0.$$

Then (4.22) follows from (4.21). As $\alpha' > \frac{1}{2}$, it follows from (4.20) that $\lim_{r \to \infty} \phi(r^2)/r = \infty$, and so $Q'(\infty) = \infty$. ∎

We shall need (4.22) in proving Theorem 1.3.

Proof of Theorem 2.6. Let θ_0 be as in the hypotheses of Theorem 2.6. We apply Lemma 4.1 with $c^* = 0$ and $l = 0$ and with

$$\phi^*(t) = \phi(t)/2, \qquad t \in [1, \infty).$$

We can assume $\phi^*(1) = 1$. Given $\varepsilon > 0$ (4.20) implies that

$$t^{\alpha' - \varepsilon} < \phi(t) < t^{\alpha' + \varepsilon}, \qquad t > t_0 = t_0(\varepsilon).$$

Hence (4.1) and (4.2) hold with $l = 0$ as $\alpha' \in (\frac{1}{2}, 1)$. Let ξ_n be defined by

$$H(\xi_n) = n, \ n \text{ large enough.}$$

From (4.20) and (4.28) with $\theta = 0$, we see

$$\begin{aligned} \lim_{n \to \infty} \log n / \log \xi_n &= \lim_{n \to \infty} \{\log H(\xi_n)/\log \phi(\xi_n)\}\{\log \phi(\xi_n)/\log \xi_n\} \\ &= \alpha', \end{aligned}$$

and (6.4) follows as $\alpha' > \frac{1}{2}$, while (4.7) follows as $H^* = (\frac{1}{2})\, H$. Further, by Lemma 3.1(v) and Lemma 4.3(iii),

$$\xi_n^{1/2} Q'(\xi_n^{1/2}) \sim Q(\xi_n^{1/2}) \sim \phi(\xi_n) \sim H(\xi_n) = n,$$

by (4.28) with $\theta = 0$. Then, as in Lemma 4.2, this implies

$$\xi_n \sim q_n^2, \qquad n \text{ large enough.} \tag{4.31}$$

Let $\{T_n\}$ be as in Lemma 4.1, and let

$$P_n(x) = T_n(x^2), \qquad n = 1, 2, \ldots.$$

by (4.4), (4.8), and (4.31), for $|x| \leqslant Cq_n$,

$$\begin{aligned} P_n(x) \sim |y(x^2 \exp(i\theta_0))| &= \exp(2\mathrm{Re}\, H^*(x^2 \exp(i\theta_0))) \\ &= \exp(-Q(x) + g(x^2)) \sim W(x), \end{aligned}$$

by (2.18). Further, by (4.9), for $|x| \leqslant Cq_n$,

$$|P_n'(x)|/W(x) \leqslant \begin{cases} C_3 H(x^2)/x & \text{if } x \geqslant 1, \\ C_3 (H(x^2)/x + 1) & \text{if } x < 1. \end{cases} \tag{4.32}$$

Finally, it is easy to see from (4.24) that

$$H(x^2)/x \to 0 \qquad \text{as } x \to 0+,$$

and as $x \to \infty$,

$$\frac{d}{dx}(H(x^2)/x) = x^{-2}H(x^2)(2x^2H'(x^2)/H(x^2) - 1)$$
$$= x^{-2}H(x^2)(2H_1'(1)/H_1(1) - 1 + o(1))$$

(by (4.28) and (4.30))

$$= x^{-2}H(x^2)(2\alpha' - 1 + o(1)) > 0,$$

by (4.26). Thus for n large, and for $|x| \leqslant Cq_n$, C small enough, the right member of (4.32) is bounded above by

$$C_3\{H(Cq_n^2)/Cq_n + 1\} \leqslant C_4 n/q_n,$$

by (4.31) and the definition of ξ_n. ∎

5. Proof of Theorem 2.4

The following lemma summarizes the results that we shall need from [9]:

Lemma 5.1. *Let $W = \exp(-Q)$ be a Freud weight, and assume that*

$$Q''(x) \geqslant 0 \qquad \text{for } x \text{ large enough.}$$

Let q_n be defined for $n \geqslant n'$, say, and let

$$G_Q(x) = 1 + \sum_{j=n'}^{\infty} (x/q_j)^{2j} j^{-1/2} \exp(2Q(q_j)), \qquad x \in \mathbb{R}. \tag{5.1}$$

Further, let $B_{2n}(x)$ be the $(n+1)$th partial sum of G_Q, so that B_{2n} has degree at most $2n$, $n = 1, 2, \ldots$. Then

(i) *G_Q is entire and*

$$G_Q(x) \sim \exp(2Q(x)), \qquad x \in \mathbb{R}. \tag{5.2}$$

(ii) *There exist C_1 and C_2 such that*

$$B_{2n}(x) \sim \exp(2Q(x)), \qquad |x| \leqslant C_1 q_n, \tag{5.3}$$

and

$$|B_{2n}'(x)| \leqslant C_2(n/q_n)\exp(2Q(x)), \qquad x \in \mathbb{R}. \tag{5.4}$$

Proof. (i) By Theorem 6(ii) in [9], G_Q is entire and

$$G_Q(x) \sim \exp(2Q(x)), \qquad x \to \infty. \tag{5.5}$$

Note that in [9, Eq. (17)], G_Q is defined in such a way that the lower index of summation is 0, rather than n' as in (5.1). However, addition or substraction of a polynomial to G_Q obviously does not affect (5.5). Next, as both G and $\exp(2Q)$ are positive and even in $\mathbb{R}$, (5.5) implies (5.2).

(ii) By Lemma 3.1(vi), there exists $C>0$ such that

$$Q(q_n) \leqslant Cn, \qquad n \geqslant n'. \tag{5.6}$$

Hence, if $|x| \leqslant \varepsilon q_n$, some $0<\varepsilon<1$,

$$\begin{aligned} |B_{2n}(x) - G_Q(x)| &\leqslant \sum_{j=n+1}^{\infty} (\varepsilon q_n/q_j)^{2j} e^{2Cj} \\ &\leqslant \sum_{j=n+1}^{\infty} (\varepsilon e^C)^{2j} < \frac{1}{2}, \end{aligned}$$

if ε is small enough. As $G_Q(x) \geqslant 1$, (5.3) follows. Next, choose a positive number $L > 2\exp(C)$ with C as in (5.6). We can write

$$\begin{aligned} |B_n'(x)| &= \sum_{j=n'}^{n} (x/q_j)^{2j} j^{-1/2} \exp(2Q(q_j))(2j/|x|) \\ &= \Sigma_1 + \Sigma_2. \end{aligned}$$

where the summation in Σ_1 ranges over $j \leqslant Ln|x|/q_n$, and Σ_2 ranges over $Ln|x|/q_n < j \leqslant n$. We see that

$$\Sigma_1 \leqslant B_n(x)(2Ln/q_n) \leqslant (2Ln/q_n)\, G_Q(x). \tag{5.7}$$

To deal with Σ_2 we note that as $Q'' \geqslant 0$, $j/q_j = Q'(q_j)$ is nondecreasing for large j. Hence, in Σ_2,

$$|x|/q_j \leqslant L^{-1}(j/q_j)(q_n/n) \leqslant L^{-1},$$

and by (5.6)

$$\Sigma_2 \leqslant 2 \sum_{1}^{n} (L^{-1} \exp C)^{2j-1} j/q_j < n/q_n \leqslant n/q_n G_Q(x). \tag{5.8}$$

Finally, (5.2), (5.7), and (5.8) yield (5.4). ∎

Proof of Theorem 2.4. Let Q and Q_1 satisfy the hypotheses of Theorem 2.4. Let M be a large positive integer and define

$$Q^*(x) = MQ(x) - (1/2)\, Q_1(x), \qquad x \in \mathbb{R}. \tag{5.9}$$

From (2.13), it follows that if M is large enough

$$Q^{*\prime\prime}(x) \sim Q''(x) > 0, \qquad x \text{ large enough.} \tag{5.10}$$

Integrating and using $Q'(\infty) = \infty$, we see that

$$Q^{*\prime}(x) \sim Q'(x) > 0, \qquad x \text{ large enough.} \tag{5.11}$$

Since $W = \exp(-Q)$ is a Freud weight, it follows from (5.10) and (5.11) that $W^* = \exp(-Q^*)$ is a Freud weight and satisfies the hypotheses of Lemma 5.1. Hence, there exist polynomials $B_{2n}(x)$ of degree at most $2n$, $n = 1, 2, \ldots$, such that for some C_1 and C_2

$$B_{2n}(x) \sim \exp(2Q^*(x)), \qquad |x| \leqslant C_1 q_n^*, \tag{5.12}$$

and

$$|B'_{2n}(x)| \leqslant C_2 (n/q_n^*) \exp(2Q^*(x)), \qquad x \in \mathbb{R}. \tag{5.13}$$

Here q_n^* is the positive root of the equation

$$q_n^* Q^{*\prime}(q_n) = n, \qquad n \text{ large enough.}$$

Now let q_n and q_n^1 respectively denote the positive roots of the equations

$$q_n Q'(q_n) \text{ and } q_n^1 Q_1'(q_n^1) = n, \qquad n \text{ large enough.}$$

From (2.12), Lemma 3.1(v) and (5.11), we deduce

$$xQ_1'(x) \sim xQ'(x) \sim xQ^{*\prime}(x), \qquad x \text{ large enough,}$$

and so Lemma 3.1(i), (ii) show that

$$q_n^1 \sim q_n \sim q_n^*, \qquad n \text{ large enough.} \tag{5.14}$$

Next, as $W = \exp(-Q)$ is regular, we can find polynomials $A_n(x)$ of degree at most n, $n = 1, 2, \ldots$, such that for some C_3 and C_4

$$A_n(x) \sim \exp(-Q(x)), \qquad |x| \leqslant C_3 q_n \tag{5.15}$$

and

$$|A'_n(x)| \leqslant C_4 (n/q_n) \exp(-Q(x)), \qquad |x| \leqslant C_3 q_n. \tag{5.16}$$

Let

$$P_n = A_n^{2M} B_{2n}, \qquad n = 1, 2, \ldots,$$

so that P_n has degree at most $(2M+2)n$. By (5.12), (5.14), and (5.15), for $|x| \leqslant C_5 q_n$,

$$P_n(x) \sim \exp(-2MQ(x)) \exp(2MQ(x) - Q_1(x)) = W_1(x),$$

while by (5.13) and (5.16),

$$\begin{aligned} |P_n'(x)| &\leqslant 2MA_n^{2M-1}(x)|A_n'(x)| B_{2n}(x) + A_n^{2M}(x)|B_{2n}'(x)| \\ &\leqslant C_6(n/q_n)\, W(x). \end{aligned}$$

Finally, to obtain polynomials of degree at most n, we can replace P_n by $P_{[n/[2M+2)]}$ and use $q_{[n/(2M+2)]} \sim q_n \sim q_n^1$. ∎

6. Proof of Theorem 2.5

The proof of Theorem 2.5 will proceed along the following lines: Assume the hypotheses of Theorem 2.5. We set

$$\phi(t) = Q(t^{1/k}), \qquad t \in [1, \infty) \tag{6.1}$$

and

$$H(r) = \int_1^\infty \frac{\phi(t)}{t+r} \left(\frac{r}{t}\right)^2 dt, \qquad r \in [0, \infty), \tag{6.2}$$

and define

$$Q^*(r) = H(r^k), \qquad r \in [0, \infty). \tag{6.3}$$

We use Theorem 2.1 to show that $W^*(r) = \exp(-Q^*(r))$ is a regular weight and then use Theorem 2.4 to show that $W(r) = \exp(-Q(r))$ is also regular.

Lemma 6.1. *Assume the hypotheses of Theorem* 2.5 *and let* ϕ, H, *and* Q^* *be as in* (6.1) *to* (6.3).

(i) *For large enough* t, $\phi(t)$ *is positive and increasing.*

(ii) *There exist* $\varepsilon > 0$ *and* C_1, C_2, C_3, *and* C_4 *such that*

$$C_1 u^{1+\varepsilon} \leqslant \phi(ut)/\phi(t) \leqslant C_2 u^{2-\varepsilon}, \qquad u \geqslant 1,\ t \geqslant C_3 \tag{6.4}$$

and

$$C_1 u^{2-\varepsilon} \leqslant \phi(ut)/\phi(t) \leqslant C_2 u^{1+\varepsilon}, \qquad 0 < u \leqslant 1,\ ut \geqslant C_4. \tag{6.5}$$

Consequently, for some C_5, C_6, *and* C_7,

$$C_5 v^{1+\varepsilon} \leqslant \phi(v) \leqslant C_6 v^{2-\varepsilon}, \qquad v \geqslant C_7. \tag{6.6}$$

(iii) $H(r) \sim \phi(r)$, *r large enough.*

(iv) $1 \leqslant rH'(r)/H(r) \leqslant 2$, $r \in (0, \infty)$.

(v) $\phi(r)/4 \leqslant r^2 H''(r) \leqslant 2H(r)$, $r \in (1, \infty)$.

Proof. (i) This follows immediately from (6.1).

(ii) From (6.1) and (2.15) we see that

$$2^k < a \leqslant \phi(2^k t)/\phi(t) \leqslant b < 2^{2k}, \qquad t \text{ large enough.}$$

Let $\lambda = 2^k$. This last inequality implies that for some $\varepsilon > 0$,

$$\lambda^{1+\varepsilon} \leqslant \phi(\lambda t)/\phi(t) \leqslant \lambda^{2-\varepsilon}, \qquad t \text{ large enough.} \tag{6.7}$$

If $\lambda^j \leqslant u < \lambda^{j+1}$, some $j \geqslant 0$, then repeated application of (6.7) shows that if $t \geqslant C_3$, say,

$$\lambda^{-(1+\varepsilon)} u^{1+\varepsilon} \leqslant \lambda^{j(1+\varepsilon)} \leqslant \phi(ut)/\phi(t) \leqslant \lambda^{(j+1)(2-\varepsilon)} \leqslant \lambda^{2-\varepsilon} u^{2-\varepsilon},$$

and (6.4) follows. Similarly (6.7) yields (6.5). Finally, fixing some large t and setting $u = v/t$, we see that (6.6) follows from (6.4).

(iii) First, for $r \geqslant 1$,

$$H(r) \geqslant \phi(r) \int_r^\infty \frac{1}{t+r} \left(\frac{r}{t}\right)^2 dt \geqslant \phi(r) \int_1^\infty (2u^3)^{-1}\, du.$$

Next, let C_4 be as in (6.5). We see that if r is large enough,

$$H(r)/\phi(r) = \left\{ \int_{1/r}^{C_4/r} + \int_{C_4/r}^{1} + \int_1^\infty \right\} \frac{\phi(ur)}{\phi(r)} \frac{1}{u+1} \frac{du}{u^2}$$

$$\leqslant \phi(C_4)(C_4 - 1)\, r/\phi(r) + C_2 \int_0^1 \frac{u^{-1+\varepsilon}}{u+1}\, du + C_2 \int_1^\infty \frac{u^{-\varepsilon}}{u+1}\, du$$

(by (6.4) and (6.5))

$$\leqslant C,$$

(by (6.6)).

(iv) This is Lemma 5.1(iii) in [6] with $l = 1$.

(v) From (2.4) and (6.2), it follows that

$$H''(r)\,r^2 = 2\int_1^\infty \frac{r^2\phi(t)}{(t+r)^3}\,dt$$
$$\leqslant 2\int_1^\infty \left(\frac{r}{t}\right)^2 \frac{\phi(t)}{t+r}\,dt = 2H(r).$$

Further

$$H''(r)\,r^2 \geqslant 2r^2\phi(r)\int_r^\infty (t+r)^{-3}dt = \phi(r)/4. \quad \blacksquare$$

Proof of Theorem 2.5. We first show that $W^* = \exp(-Q^*)$ is regular. Now

$$Q^{*\prime}(r) = kr^{k-1}H'(r^k) \sim \phi(r^k)/r, \qquad r \text{ large enough}, \tag{6.8}$$

by Lemma 6.1(iii) and (iv). Further, by Lemma 6.1(iii), (iv), and (v),

$$Q^{*\prime\prime}(r) = (kr^{k-1})^2\,H''(r^k) + k(k-1)\,r^{k-2}H'(r^k)$$
$$\sim \phi(r^k)/r^2, \qquad r \text{ large enough}. \tag{6.9}$$

It follows that W^* is a Freud weight. From (6.2) and (6.3) we see that Q^* admits a representation of the form (2.2), while (2.3) follows from (6.6). Hence W^* is regular.

Next, (6.1), (6.3), and Lemma 6.1(iii) show that

$$Q(x) \sim Q^*(x), \qquad x \text{ large enough},$$

while by (1.2) and Lemma 3.1(v),

$$|Q''(x)| \leqslant (B+1)\,Q'(x)/x \leqslant CQ(x)/x^2$$
$$= C\phi(x^k)/x^2$$
$$\leqslant C_1 Q^{*\prime\prime}(x),$$

by (6.9). Hence Theorem 2.4 shows that $W = \exp(-Q)$ is also regular. $\blacksquare$

7. Proof of Theorem 1.3

The remarks after Theorem 2.5 show that Theorem 1.3 is true if in (1.8), $\alpha > 2$, but $\alpha \neq 4$. Hence, we assume below that $1 < \alpha < 2$. Assuming the hypotheses of Theorem 1.3, let

$$\phi(t) = Q(t^{1/2}), \qquad t \in (1, \infty),$$
$$\alpha' = \alpha/2, \tag{7.1}$$

and choose $\theta_0 \in (-\pi, \pi)$ such that

$$\cos(\alpha'\theta_0) < 0. \tag{7.2}$$

Define

$$H(z) = \int_1^{\infty} \frac{\phi(t)}{t+z}\left(\frac{z}{t}\right) dt, \qquad z \in \mathbb{C}\backslash(-\infty, 0). \tag{7.3}$$

Finally, define Q^* by

$$Q^*(r^{1/2}) = -\operatorname{Re}\{H(r\exp(i\theta_0))\}, r \in (0, \infty). \tag{7.4}$$

We shall use Theorem 2.6 to show that $W^*(x) = \exp(-Q^*(x))$ is regular, and then use Theorem 2.4 to show W is regular. It follows from Lemma 7.1(iii) below that H is well defined.

LEMMA 7.1 *Assume the hypotheses of Theorem* 1.3 *with* $1 < \alpha < 2$, *and let* ϕ, θ_0, H, *and* Q^* *be as in* (7.1) *to* (7.4).

(i) $\phi(t)$ *is positive and* $\phi'(t) > 0$ *for large* t.

(ii) $\lim_{t\to\infty} t\phi'(t)/\phi(t) = \alpha'$.

(iii) *Let* $\varepsilon > 0$. *For* t *large enough,*

$$t^{\alpha'-\varepsilon} \leqslant \phi(t) \leqslant t^{\alpha'+\varepsilon}.$$

(iv) $Q^{*(j)}(r) \sim \phi(r^2)/r^j$, $j = 0, 1, 2$; r *large enough.*

Proof. (i), (ii) These follow immediately from (1.8) and (7.1).

(iii) This follows from (ii).

(iv) First note that, by definition, Q^* admits a representation of the form (2.18) with $g(r) \equiv 0$. Further, we have shown that (2.19) and (2.20) are true, while (7.2) implies (2.21). Then Lemma 4.3(iii) and (iv) yield the result. ∎

Proof of Theorem 1.3 *when* $1 < \alpha < 2$. First, it follows easily from Lemma 7.1(iv) that $W^* = \exp(-Q^*)$ is a Freud weight. As already discussed, Q^* admits a representation (2.18), and so $W^* = \exp(-Q^*)$ is regular. Further, $Q^{*\prime}(\infty) = \infty$, by Lemma 4.3(iv).

Next, $Q(x) \sim Q^*(x)$ for large x, by (7.1) and Lemma 7.1(iv). Finally, as in the proof of Theorem 2.5, one sees that $|Q''(x)| \leqslant CQ^{*\prime\prime}(x)$ for large x, and so W is also regular by Theorem 2.4. ∎

REFERENCES

1. N. I. AKHIEZER, "The Classical Moment Problem and Some Related Questions in Analysis," Oliver & Boyd, Edinburgh, 1965.
2. R. BOAS, "Entire Functions," Academic Press, New York, 1954.
3. S. BONAN AND P. NEVAI, A Markov–Bernstein theorem, manuscript.
4. G. FREUD, On polynomial approximation with the weight $\exp(-x^{2k}/2)$, *Acta Math. Hungar.* **24** (1973), 363–371.
5. G. FREUD, On Markov–Bernstein type inequalities and their applications, *J. Approx. Theory* **19** (1977), 22–37.
6. A. L. LEVIN AND D. S. LUBINSKY, Canonical products and the weights $\exp(-|x|^\alpha)$, $\alpha > 1$, with applications, *J. Approx. Theory* **49** (1987), 149–169.
7. D. S. LUBINSKY, A weighted polynomial inequality, *Proc. Amer. Math. Soc.* **92** (1984), 263–267.
8. D. S. LUBINSKY, Estimates of Freud–Christoffel functions for some weights with the whole real line as support, *J. Approx. Theory* **44** (1985), 343–379.
9. D. S. LUBINSKY, Gaussian quadrature, weights on the whole real line and even entire functions with nonnegative even order derivatives, *J. Approx. Theory* **46** (1986), 297–313.
10. P. NEVAI, Some properties of polynomials orthonormal with weight $(1+x^{2k})^\alpha \exp(-x^{2k})$ and their applications in approximation theory, *Soviet Math. Dokl.* **14** (1973), 1116–1119.